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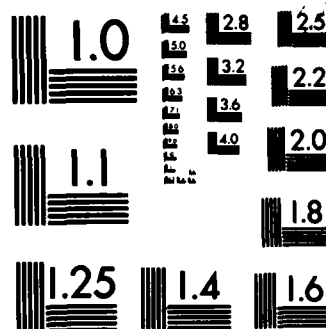
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INTRODUCTION

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Chief, Technical Information Division

The moving finite element (MFE) method is a new approach for numerically solving partial differential equation (PDE) systems;^(1,2,3) it is particularly well suited for resolving PDE solutions which may contain large, multiple gradients over highly disparate scales in both space and time. These types of PDE's abound in such basic technical disciplines as aerodynamics (with emphasis on shear layers, shocks and their possible interactions), combustion, plasma physics, interface phenomena, continuum mechanics, and other transport processes.

In the MFE method, grid co-ordinates themselves are dependent variables which are calculated continuously at each time step in order to minimize PDE residuals. This feature has successfully suppressed numerical dissipation to very low levels and has resolved accurately in 1-D those physical transport processes which may occur over extremely small scales (e.g. those described by the Navier-Stokes equations for viscous compressible fluids) simultaneously with other physical processes which may span macroscopic scales. The objectives of the present research effort are to extend the MFE method to 2-D and to investigate its basic properties and needs for solving important PDE's in 2-D. The work tasks which are contributing to the achievement of these objectives are discussed in the next section of this report.

It was established during the first year of this study that the MFE method does extend logically and practically to 2-D. An initial experimental 2-D MFE code version was developed in that early work for the purpose of conducting continuing theoretical and applied mathematical research in diverse scientific contexts. Also in the first year of study, effective MFE node movement properties and significant node savings were demonstrated for simple -- but yet significant -- test problems in 2-D. The MFE code structure was found to be amenable to vectorization and to use on envisioned advanced computers.

Having established these milestones in the first year, work during this second year of study has moved to a new level, focussing now on a host of those issues and research needs which are essential to the use of the MFE

method in truly large-scale 2-D computations. As in the past, some of the emerging MFE results in 2-D may be unique and/or contrary to conventional wisdoms. In several areas, they suggest some important lines of research which should perhaps be pursued more intensively in the future, both by ourselves and by others.

This report is organized as follows: the Technical Progress section provides a substantive statement of significant accomplishments and progress toward achieving the research objectives. The Publications section provides a cumulative chronological list of written publications in technical journals and includes those in press and manuscripts of articles in preparation. The Personnel section lists the professional personnel associated with the research effort, and the final section entitled Interactions lists coupling activities.

I. TECHNICAL PROGRESS

Work during this period addressed, among other tasks, MFE node movement properties, ODE solvers for PDE methods, linear systems solvers for the MFE method, regularization techniques for the MFE method, boundary conditions, alternative co-ordinate systems, and interface tracking. This was essentially a building year; i.e., one of building, testing, and implementing both new and existing mathematical methods which are needed in the investigation of the MFE method in 2-D. In several areas above our MFE work has barely opened entire new lines of inquiry, and results are either preliminary or incomplete. In such cases, the discussion of progress is necessarily general. On the other hand there are some significant implications which are already apparent in some of the new research areas which have been broached during this period for which the potential significance is indicated more specifically -- but with the understanding that a great deal of additional work is needed in order to further substantiate or refute the early indications.

1. MFE Node Movement Properties

Burger's test example can be formulated to pose computational challenges of varying degrees of difficulty to PDE methods in 2-D. This can be done by

selecting initial conditions which give rise not only to steep shocks with planar profiles (as in work during year 1 of this study) but also to highly skewed wave profiles with gradients of greater or lesser magnitudes.

The PDE's for this skewed Burger's model problem are given by:

$$u_t = -uu_x - vu_y + v(u_{xx} + u_{yy}) \quad (1)$$

$$v_t = -uv_x - vv_y + v(v_{xx} + v_{yy}) \quad (2)$$

where u is the x component of velocity and v is the y component, and v is an effective diffusion coefficient. Shocks are generated with gradient magnitudes on the order of $1/v$. Initial conditions which produce a doubly skewed wavefront profile are shown schematically in Figures 1 and 2. (The counterposed initial velocity fields are designed to create an evolving shock profile which is skewed in both the x and y components of velocity.) Boundary conditions are given by:

$$\begin{array}{ll} u(0,y) = u(1,y) = 0 & 0 \leq y \leq 1. \\ v_x(0,y) = v_x(1,y) = 0. & 0 \leq y \leq 1. \\ u(x,1) = 0.2 \sin \pi x & 0 \leq x \leq 1. \\ u(x,0) = -.2 \sin \pi x & 0 \leq x \leq 1. \\ v(x,1) = -1. + 0.2 \cos \pi x & 0 \leq x \leq 1. \\ v(x,0) = 1. + 0.2 \cos \pi x & 0 \leq x \leq 1. \end{array}$$

The MFE nodes are fixed by zero Dirichlet conditions along the top and bottom horizontal edges of the domain. The nodes are free to move vertically by symmetric boundary conditions along the left and right edges of the domain.

This problem can be solved readily by perhaps many PDE solution methods whenever v assumes sufficiently large values. For example, a value of $v = 0.02$ produces shock gradients on the order of 10^2 .

The MFE method requires only an 8×8 grid to give reasonably accurate solutions to this problem, and Figures 3 and 4 show very accurate MFE solutions on a 12×12 grid. Here Figure 3 presents an isometric view of the

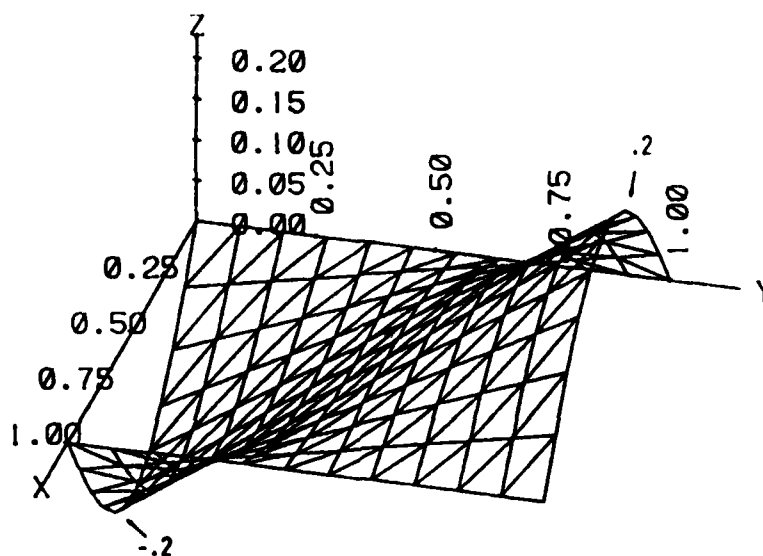


Figure 1. Plot of initial values of u in the 2-D Burger-like example on a 12×12 grid mesh.

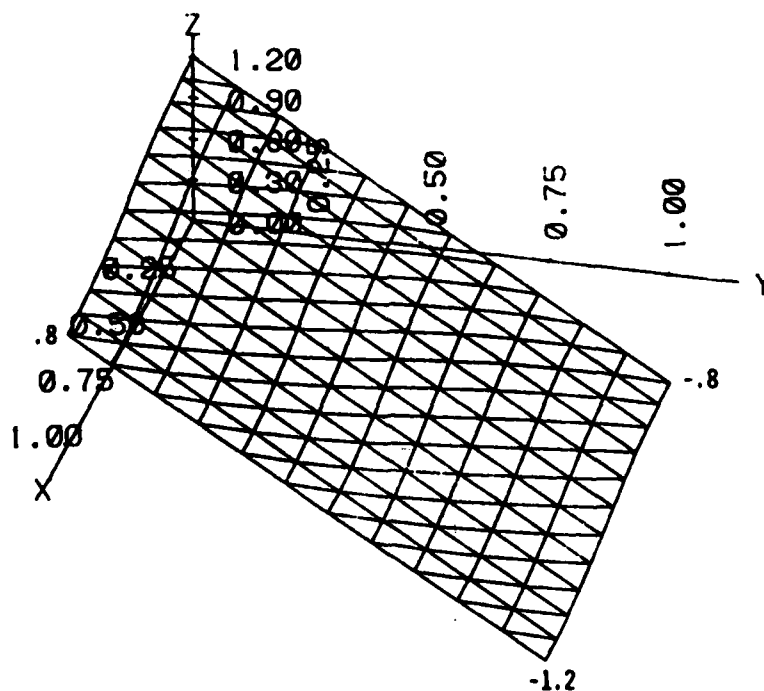


Figure 2. Plot of initial values of v in the 2-D Burger-like example on a 12×12 grid mesh.

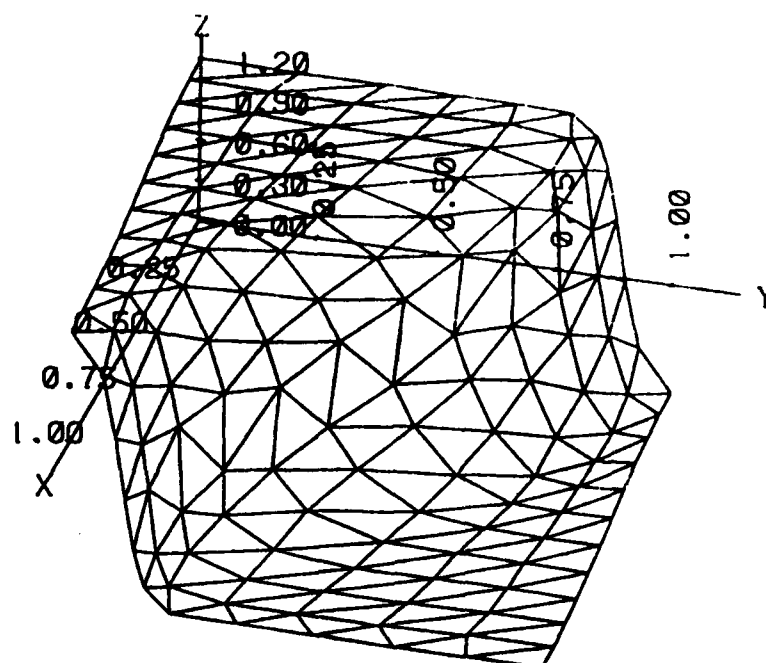


Figure 3. Isometric view of v at $t = 3.0$ in the 2-D Burger-like example on a 12×12 MFE grid.

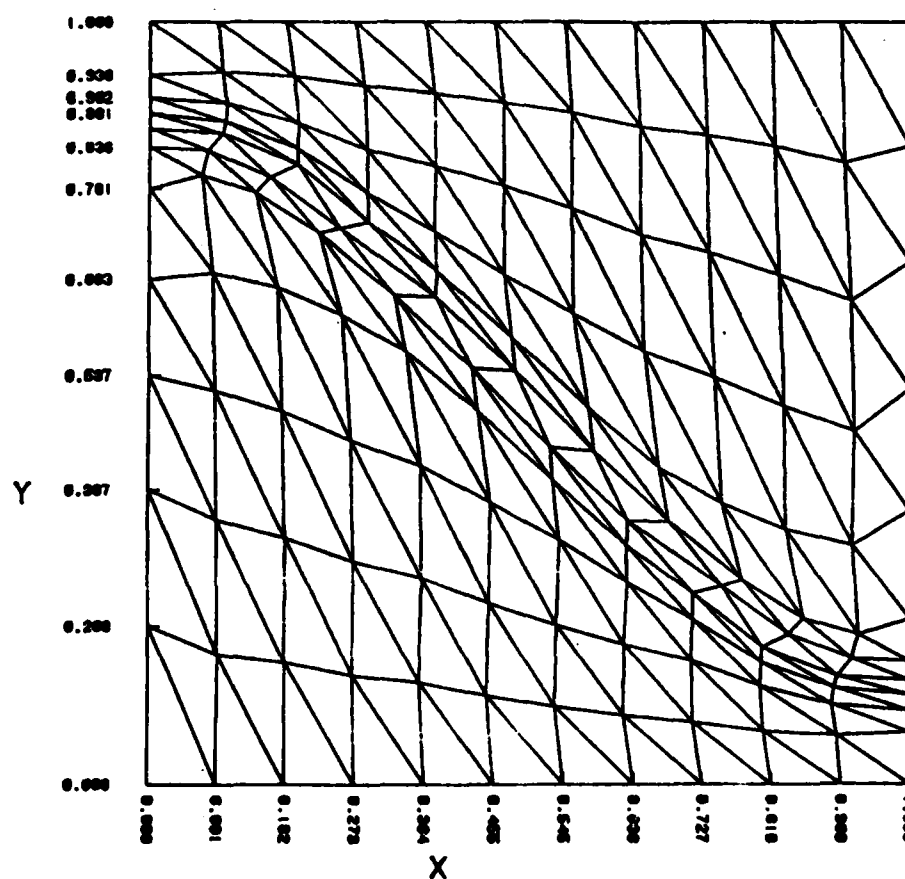


Figure 4. MFE grid projections on the X-Y plane at $t = 3.0$ in the 2-D Burger-like example on a 12×12 grid.

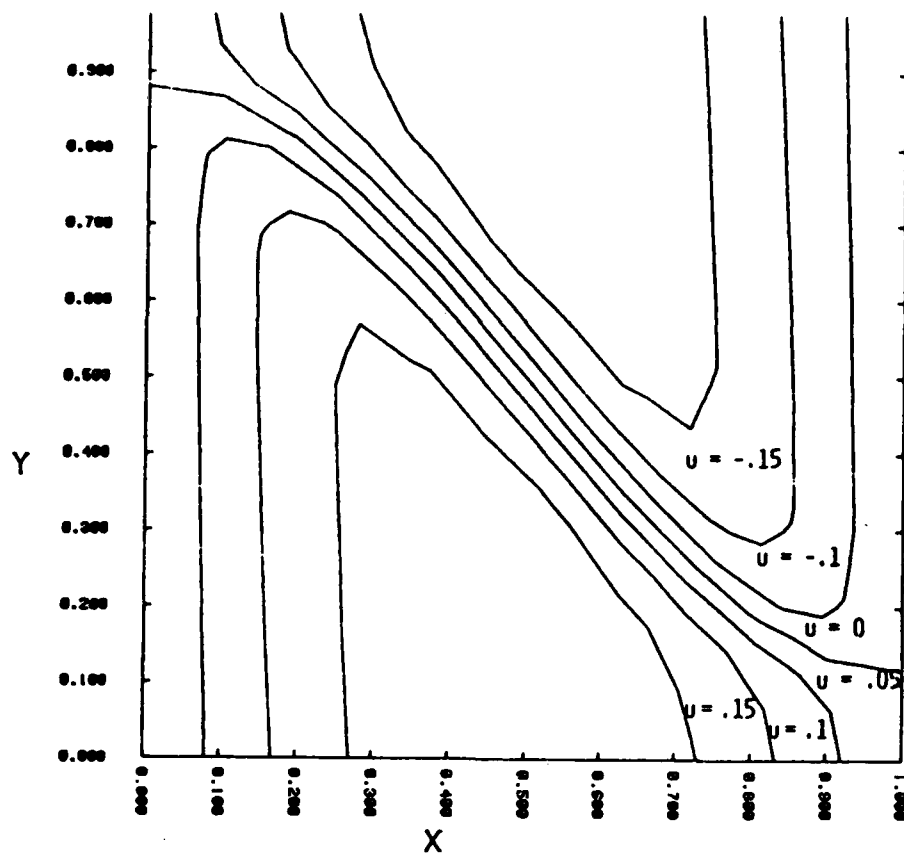


Figure 5. Contour plots of selected values of u at $t = 3.0$ in the 2-D Burger-like example on a 12×12 grid.

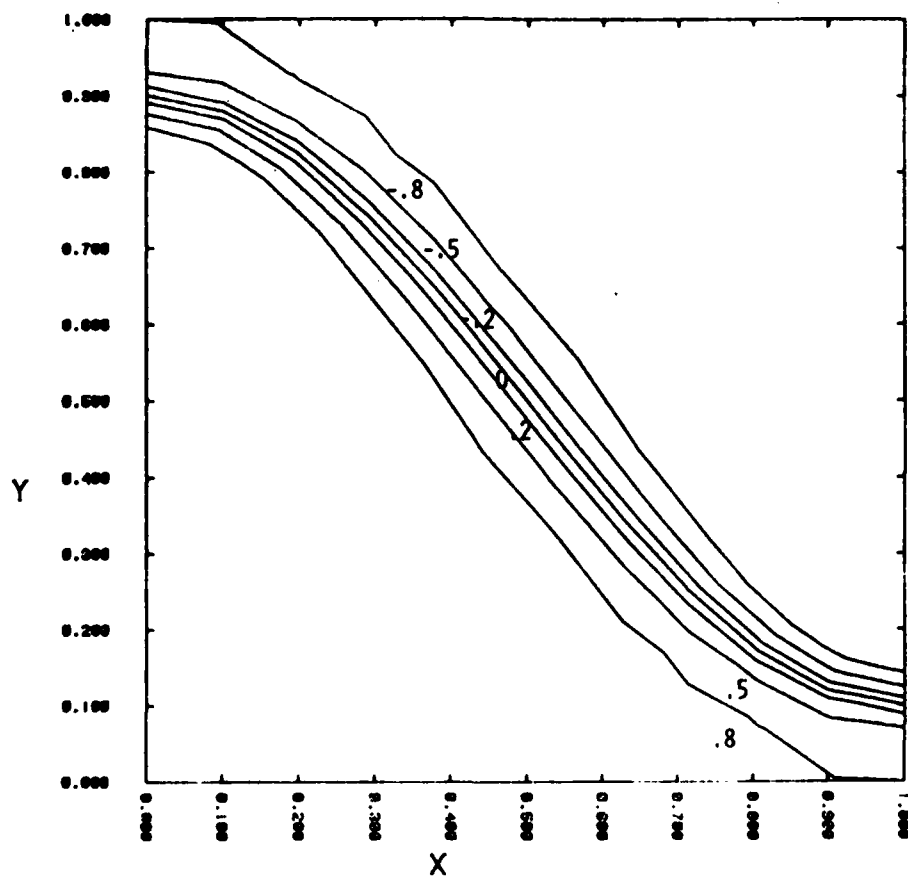


Figure 6. Contour plots of selected values of v at $t = 3.0$ in the 2-D Burger-like example on a 12×12 grid.

evolving profile of the y component of velocity at $t = 3.0$, well after the shock has formed and after the wavefront has undergone significant shearing. The x-component of velocity is sufficiently sheared that a hidden line plot, which is not yet available, is required for easy interpretation by the naked eye. The MFE grid nodes have migrated extensively from their initial positions as can be seen in Figure 4 which represents the grid mesh projected onto the x-y plane at $t = 3.0$. Figures 5 and 6 present contour plots for selected constant values of u and v , respectively, at $t = 3.0$. It is evident from the magnitudes of shock gradients and from the regions of significant curvature which span nearly the entire domain that an alternative PDE method with a fixed grid may require on the order of 10^4 , or more, grid nodes in order to achieve comparable degrees of accuracy in this problem.

This same basic problem can now be made to correspond to a much more demanding physical problem by letting $\nu = 0.002$. Figure 7 shows an isometric view of the MFE solution on a 16×16 grid for this case. Shock gradients are now generated with magnitudes of several times 10^3 . Before discussing these MFE results in detail, some general observations should be discussed: It is extremely unlikely that any other existing PDE method using either a fixed grid or a less than optimal adaptive grid can accurately solve this test problem with fewer than 10^5 - 10^6 grid nodes. It should also be noted here that numerous inviscid solvers which are under development do not apply at all to this type of advection-diffusion problem because the Laplacian is an essential mathematical operator whose effects must be rigorously resolved in advection-diffusion PDE's. Because inviscid solvers do not generally solve PDE's which contain Laplacians, they generate shocks with gradient shapes and magnitudes that are governed exclusively by the selected gridding and/or by the purely numerical dissipative processes in the inviscid method, per se. Consequently, inviscid solvers have no chance of resolving correctly any of those physical dissipation effects which are usually expressed by Laplacian operators and are present with fundamental physical significance in transport theory, hydrodynamics, plasma physics, continuum mechanics, and many other disciplines in the physical sciences. This critical discussion is not intended to denigrate the extensive research efforts on inviscid PDE solvers and/or fixed node PDE methods; but it does suggest that efforts to accommodate Laplacian operators in otherwise inviscid solution methods and efforts

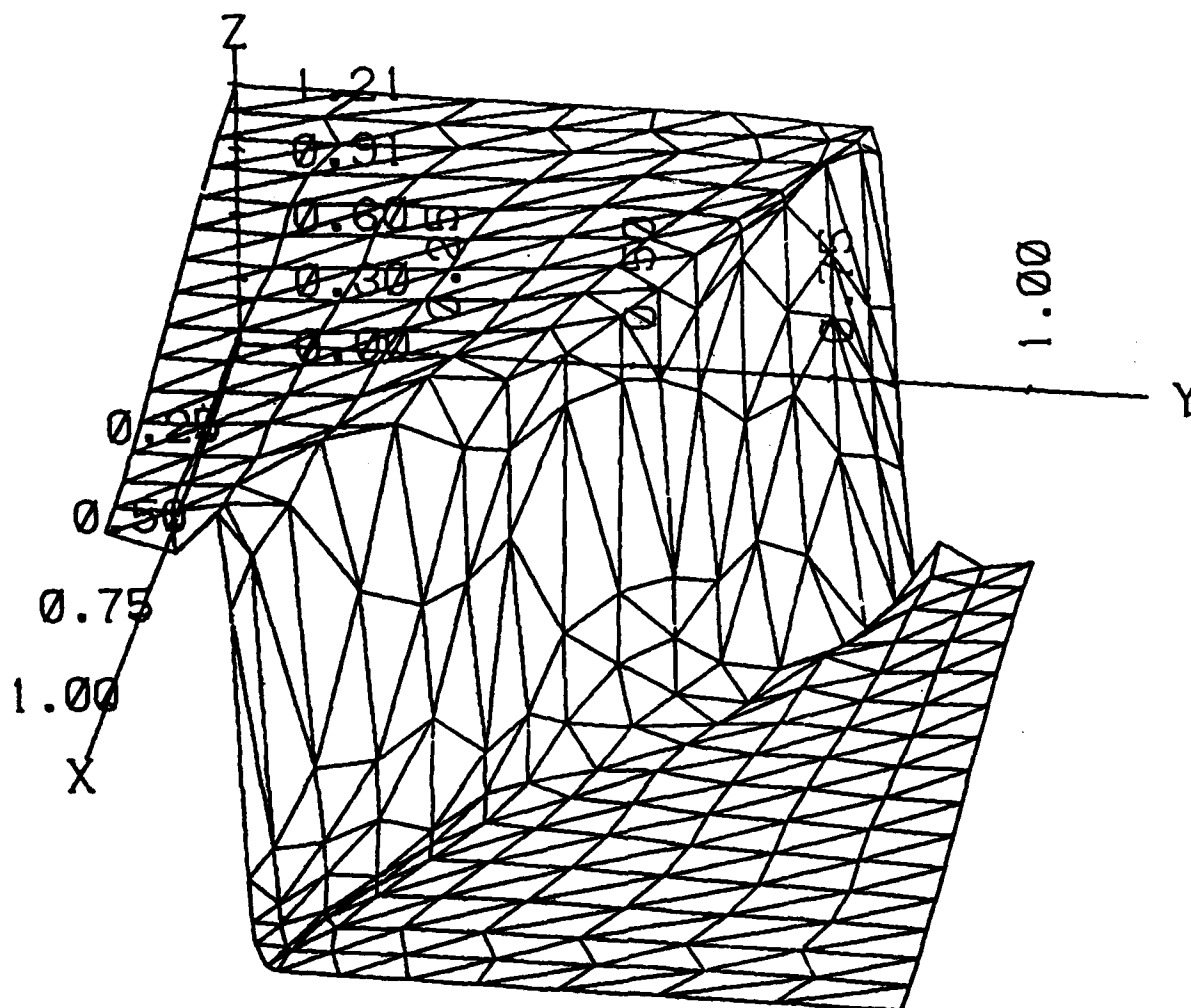


Figure 7. Isometric view of Burger's test example at $t = 1.8$ with $\nu = 0.002$ on a 16×16 MFE mesh. Shock gradients have magnitudes of approximately 10^3 in this solution for u .

to investigate more optimal adaptive grid methods for use in many existing PDE methods which are applied to advection-diffusion problems should now assume greatly increased significance. In the meantime, the MFE method is proving to be a certain kind of research pacesetter, and it is providing various clues to some of the significant new areas where mathematics research can profitably be intensified, as will be discussed further in other task areas below.

Early MFE results in 2-D are apparently continuing the trend which appeared in previous 1-D results. There, MFE solutions of both the Navier-Stokes and physically dissipative continuum mechanics equations in 1-D exhibited perhaps unprecedented simultaneous resolution of extremely disparate microscale and macroscale physical processes.^(4,5) While 2-D MFE results which emerged during this period exhibit similar promising features, numerous mathematical problems still require resolution in order to attain fully the desired levels of success in truly large-scale problems in 2-D. Clues to these problem areas can be seen in Figure 7.* For example, the irregularity of the grid triangles in the face of the shock could eventually prove to be troublesome. Similarly, the small oscillation at the base of the shock in this run is unsatisfactory, even though it can be eliminated in any number of ways. Extensive testing and analysis has indicated that the causal mechanisms underlying such mesh irregularities and oscillations in 2-D can be associated with: (i) time step and error control policies in the basic ODE integrator of Gear which is presently used, (ii) convergence properties of the linear solver, and (iii) limitations in the first-generation regularization functions. Each of these areas has been under intensive investigation during this period and some early results and their implications are discussed below.

2. ODE SOLVERS FOR PDE METHODS

The current status in this task area is that most existing ODE solvers are not well-suited for ready implementation in either the MFE method or numerous other advanced PDE methods. This critical comment is, again, not intended to denigrate the impressive advances in ODE research and development during the past decade; instead, it is intended to bring a strong new focus upon the needs of PDE solution methods, in general, and more specifically upon the pressing needs of adaptive grid PDE methods which may involve large numbers of discretized equations with highly distorted grids. Large distorted grid meshes may, in turn, augur for iterative linear solvers which can solve poorly conditioned matrix equations, as will be discussed further below.

*It is apparent that these suggested mathematical problems will have to be resolved not only for the MFE method but also for most other advanced PDE methods which may seek to solve the difficult advection-diffusion equations which frequently arise in physical problems.

A basic difficulty with most stiff ODE software has come to the fore in the present MFE research; i.e., most ODE solver packages have been designed to accommodate many different types of classic ODE problems. By classic, one refers here to such problems as chemical kinetics systems in which the dependent variables (e.g. species concentrations) are all generically the same. The error and time step controlling policies in solvers for classic ODE problems are usually less than satisfactory for applications of the ODE package to PDE solution methods. In PDE systems, the spatial dependence of generically dissimilar variables comes into play. In fluid dynamics problems, for example, the overall array of PDE variables which have been discretized on N grid nodes (x_1, x_2, \dots, x_N) can be represented as $\{\rho_1, m_1, E_1, \rho_2, m_2, E_2, \dots, \rho_N, m_N, E_N\}$, where $\rho_1 \equiv \rho(x_1)$, $\rho_2 \equiv \rho(x_2)$, etc. An ODE solver then operates on this array of discretized PDE variables as a single large vector $\{y_1, y_2, y_3, y_4, \dots, y_{3N-2}, y_{3N-1}, y_{3N}\}$, where $y_1 = \rho_1$, $y_2 = m_1$, $y_3 = E_1$, $y_4 = \rho_2$, etc. Because the error control policies in the Gear ODE package, for example, are based upon L^2 norm of all normalized quantities $y_i/(y_i)_{\max}$ unacceptably large errors can be admitted in some individual components of ρ , m , or E at arbitrary spatial locations. A much better measure for error control policies in PDE applications are maximum norms applied to each discretized PDE variable. The implementation of alternative norms is found to extend deeply into the logical structure of most ODE software packages, and alterations must usually be performed by someone who is intimately familiar with the ODE package. (Dr. Said Doss is the local expert on these matters in our MFE research.)

In view of such considerations, significant levels of effort have been devoted during this work period to: (i) modifications of Gear's basic ODE method for MFE computations; this involved wholesale alterations of the internal Gear code structure and also extensive considerations of scaling of MFE problem variables; and (ii) inevitably, the development of entirely new ODE integration procedures which better serve PDE solution needs.

The extensive modifications to Gear's method have sufficed to solve moderately challenging PDE's with modest numbers of MFE grid nodes as was seen above. But it is now clear, also, that completely new ODE code structures will be needed in pending large-scale MFE computations. We have, therefore, undertaken the development of a low-order Runge-Kutta integration

package for MFE computations. This solver addresses several PDE needs: First, error control measures operate on flexibly ordered variable arrays using maximum norms on a (PDE) component-by-component basis. Second, PDE solutions have been found to require much more gradual time step advancement policies than have been built into most classic ODE solvers.* This distinction between classic ODE and PDE time step properties apparently stems from the fundamentally coupled space-time dependences in PDE systems, vis à vis classic ODE problems which have no direct or implied spatial dependences. Whereas it is computationally worth the effort to attempt very large incremental increases (sometimes by several orders of magnitude) in Δt in classic ODE applications --even if such attempts may sometimes fail--one finds that the computational penalties for unsuccessful large Δt increases are much more severe in those PDE applications where space-time couplings augur intrinsically for more gradual Δt advancement policies. Third, time step control policies in the new ODE solver also incorporate convergence criteria from iterative linear systems solvers. Such iterative solvers should henceforth be used in large-scale MFE computations in the interest of minimizing computer memory requirements. Fourth, low-order ODE methods are now used because high-order solvers provide no apparent advantages in MFE applications and because low-order methods simplify the numerical logic, improve the code reliability, and avert possible errors associated with changes of order which are sometimes present in classic ODE system solvers. Finally, constraints on allowable fractional changes in PDE dependent variables are incorporated in the overall time step control policy in the new ODE solver. This new ODE solver is presently being implemented for use in large-scale MFE computations; and detailed descriptions of this solver, in conjunction with MFE test applications, will appear in forthcoming journal submissions.

From these initial results it is clear that renewed ODE research efforts on PDE-related problems, from several possible conceptual bases, is now timely if not long overdue. Our own efforts have barely begun to uncover many of the most pressing needs, much less to perform the extensive detailed tasks of numerical analysis which should now be pursued. Certainly, the present MFE project would benefit from expanded ODE efforts which deal with: (i) scaled

*These policies also extend deeply into the ODE code structure.

and unscaled systems of incommensurate variables on arbitrarily connected grid meshes, (ii) linkage of certain Δt -sensitive convergence criteria into the general ODE integration control policy, and (iii) splitting of the solution of the grid node equations from the solution of the discretized equations for the physical variables.

3. LINEAR SOLVERS FOR THE MFE METHOD

Advection-diffusion equations have steadfastly resisted (if not defied) satisfactory numerical solution whenever they have been used to describe physical processes over highly disparate scales. Such problems occur, for example, in numerous applications of Navier-Stokes equations to viscous compressible fluids which may contain shear layers, shocks, and separated flows. The basic difficulty derives from the nature of the matrix equations which must be solved in numerical PDE methods that are applied to these problems. The matrix equations for discretized advection-diffusion PDE's are large, sparse linear systems in which the matrices are non-symmetric and are not dominated by terms on the diagonal. The skewness of these PDE matrices can become quite large for large Δt 's and for highly distorted grid meshes, both of which are key factors in efficient solutions of these types of advection-diffusion equations.

The simple advection-diffusion equation, $y_t + \underline{c} \cdot \underline{\nabla} y = \nu \nabla^2 y$, can be used to illustrate some of these features. Upon discretization of the advection-diffusion equation, a linear system of the form $(A+B)X = C$ must be solved. The matrix A represents the stencil associated with the advection operator, and the matrix B represents the stencil associated with a nine point difference scheme for the Laplacian operator in 2-D. For representative values of Δt , these matrices may contain elements with the scaled magnitudes shown below:

$$A = \begin{pmatrix} 0 & 15 & 0 \\ -15 & 0 & 15 \\ 0 & -15 & 0 \end{pmatrix} \quad (3)$$

$$B = \begin{pmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{pmatrix}, \quad \text{and} \quad (4)$$

$$(A+B) = \begin{pmatrix} 1 & 19 & 1 \\ -11 & -20 & 19 \\ 1 & -11 & 1 \end{pmatrix}. \quad (5)$$

Testing and analysis has revealed that most available linear solvers have relatively poor rates of convergence when such significant large elements can occur away from the diagonal in non-symmetric matrices. For example, such iterative matrix solution methods as conjugate gradient, multi-grid and numerous other modern linear systems solvers which work well for symmetric matrices in discretized elliptic equations and/or for uniform grid meshes do not converge satisfactorily in presently considered advection-diffusion problems. The source of difficulty for the existing linear solvers clearly derives from the highly skewed matrices and their off-diagonal dominance. We have also shown during this work period that the direct L-U decomposition method which has been used in the Gear method until recently becomes both noisy and computer storage limited when large bandwidths arise in problems with more than a moderate number of MFE grid nodes. Again, we do not wish to denigrate the extensive ongoing work on linear systems solvers--but, rather, to call attention to these essential keys to further progress in certain PDE research areas.

Having identified more clearly the significance of these issues, we have developed in collaboration with Professor Keith Miller one promising new approach to handling more effectively these imposing PDE requirements on linear systems solvers. This new matrix solution scheme has, so far, achieved good convergence rates for Δt 's which may be 10 to 20 times greater than the large values of Δt called for by the ODE integrator. (It is generally hoped in PDE solutions that the time step size is determined by the truncation error of the ODE integrator and not by severely limited convergence properties of the linear solver.) This advanced linear solver has solved the Burger's equations discussed above with the same CPU cost as the direct L-U decomposition method in the Gear solver, but with greatly reduced storage requirements. Implementation of this new linear solver for large-scale MFE computations is progressing well, and details of this new approach will appear in journal submissions during the next work period. Clearly, our initial efforts on a more adequate linear solver for advection-diffusion problems have only barely opened a new facet of research which now warrants much more intensive theoretical and practical analysis, both by ourselves and by many others.

4. REGULARIZATION

Regularization techniques have rarely been used systematically, if at all, in PDE research in the past. There is thus presently great confusion and misunderstanding of the role of regularization techniques in PDE solution methods. On the one hand, many practitioners of conventional PDE methods suspect that regularization is an unfair trick by which one can force PDE solutions to come out in any desired manner and that such methods can therefore not be trusted. On the other hand, regularization methods are proving to be valid and powerful mathematical tools which can now be applied to achieve effective grid movement criteria systematically and to ensure that high PDE solution accuracy is also achieved in the process.

Only the simplest, first-generation regularization functions have been used in 2-D MFE problems to date. These penalty functions act like springs and/or dashpots in their action on mesh triangle altitudes (see Figure 8). The current penalty functions allow mesh triangles to distort more or less arbitrarily, so long as altitude magnitudes remain positive and maintain some designated minimum separation. This simple strategy has worked remarkably well in a surprisingly broad range of 2-D problems, including all of the results shown in this report. We are presently probing the limits of adequacy of this simple first-generation regularization method in such extended applications as Mach stems in gas dynamics and dynamic impact problems in continuum mechanics. In the Burger's test example discussed above with $v = 0.002$ and a 16×16 MFE mesh, minimum nodal separations were chosen to be much smaller than v . As seen in earlier figures, the grid triangles can become very irregular and assume configurations with extremely large aspect ratios ($O(10^2)$ to $O(10^3)$). Figures 9 and 10 below show this same Burger's test problem run with "softer" dashpot forces acting on triangle altitudes than in the run shown previously in Figure 7. The triangles in the face of the shock are more regular in this latter run, and their compaction in the center of the wavefront resolves the region where intense shearing occurs. Figure 11 shows the projection of the MFE grid mesh on the x-y plane in this example. Selected mesh connections have been traced in heavy ink in these latter figures in order to indicate the general migration pattern of the MFE nodes. It is evident that extremely large mesh triangle aspect ratios have

In the long run, there is no fundamental reason, or desire, for the MFE mesh to always be as highly skewed as the physical flow lines in order to accurately resolve such shear layers. (It is nevertheless encouraging at this stage of development that the mathematical potency of the MFE method is sufficiently great to handle such large grid aspect ratios effectively.) Alternative regularization criteria which would promote mesh homogeneity (e.g., by minimization of grid triangle aspect ratios) are presently under consideration for use in conjunction with the first-generation regularization functions. Several possible formulations of grid homogenizing regularization schemes will be tested in the next work period. These new criteria are expected to improve numerical conditioning properties and thus lead to greater computational economy in MFE codes. Mesh homogeneity should also be a significant factor in applications which must resolve turbulent eddies and/or other rotating flows.

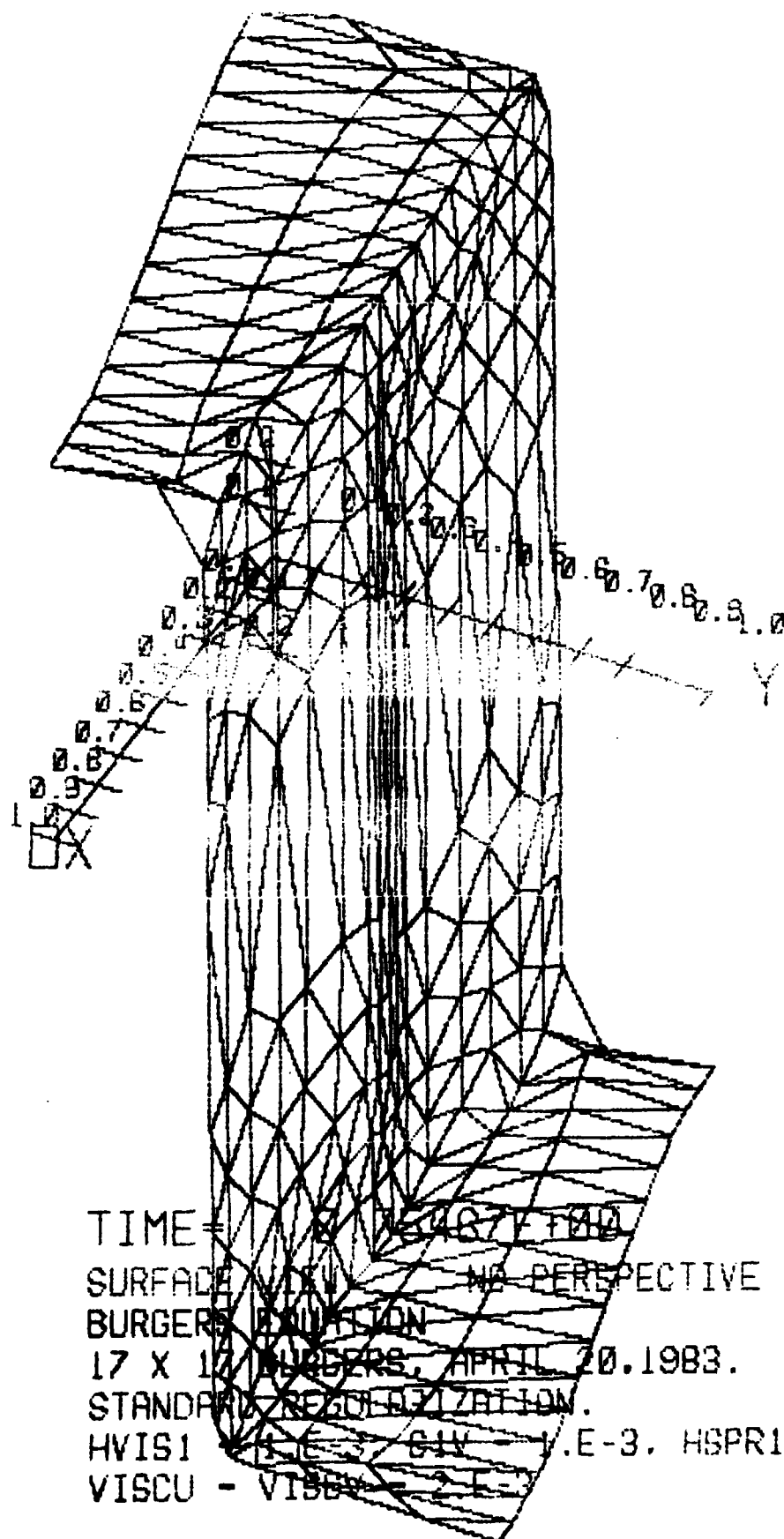


Figure 9. Isometric view of the 2-D solution of the velocity component, v , in the Burger-like example with $\nu = 0.002$ on a 16×16 MFE grid.

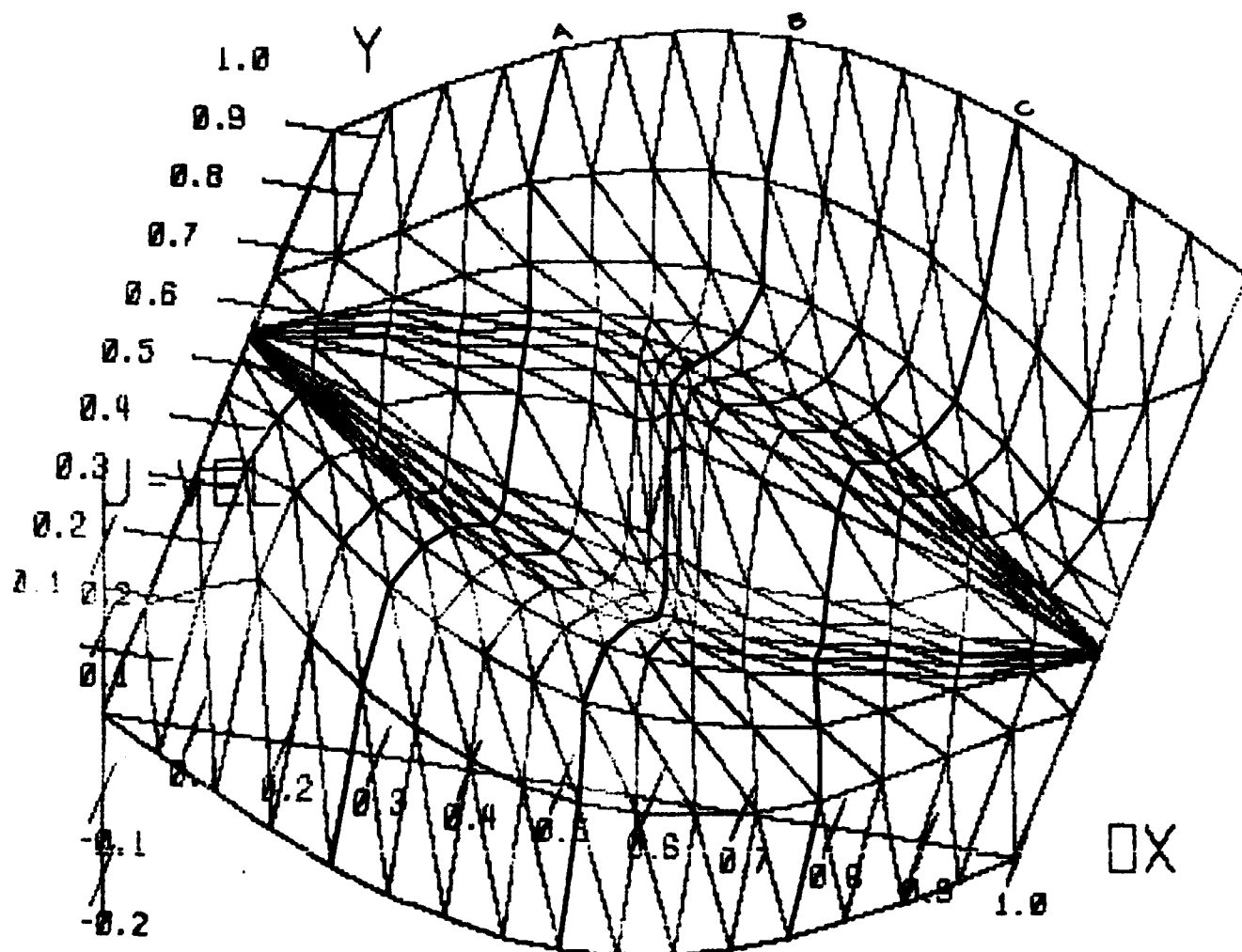


Figure 10. Isometric view of the 2-D solution of the velocity component, u , in the Burger-like example with $\nu = 0.002$ on a 16×16 MFE grid. Note that the viewing angle is rotated by 90° for a clearer view of the doubly skewed wavefront.

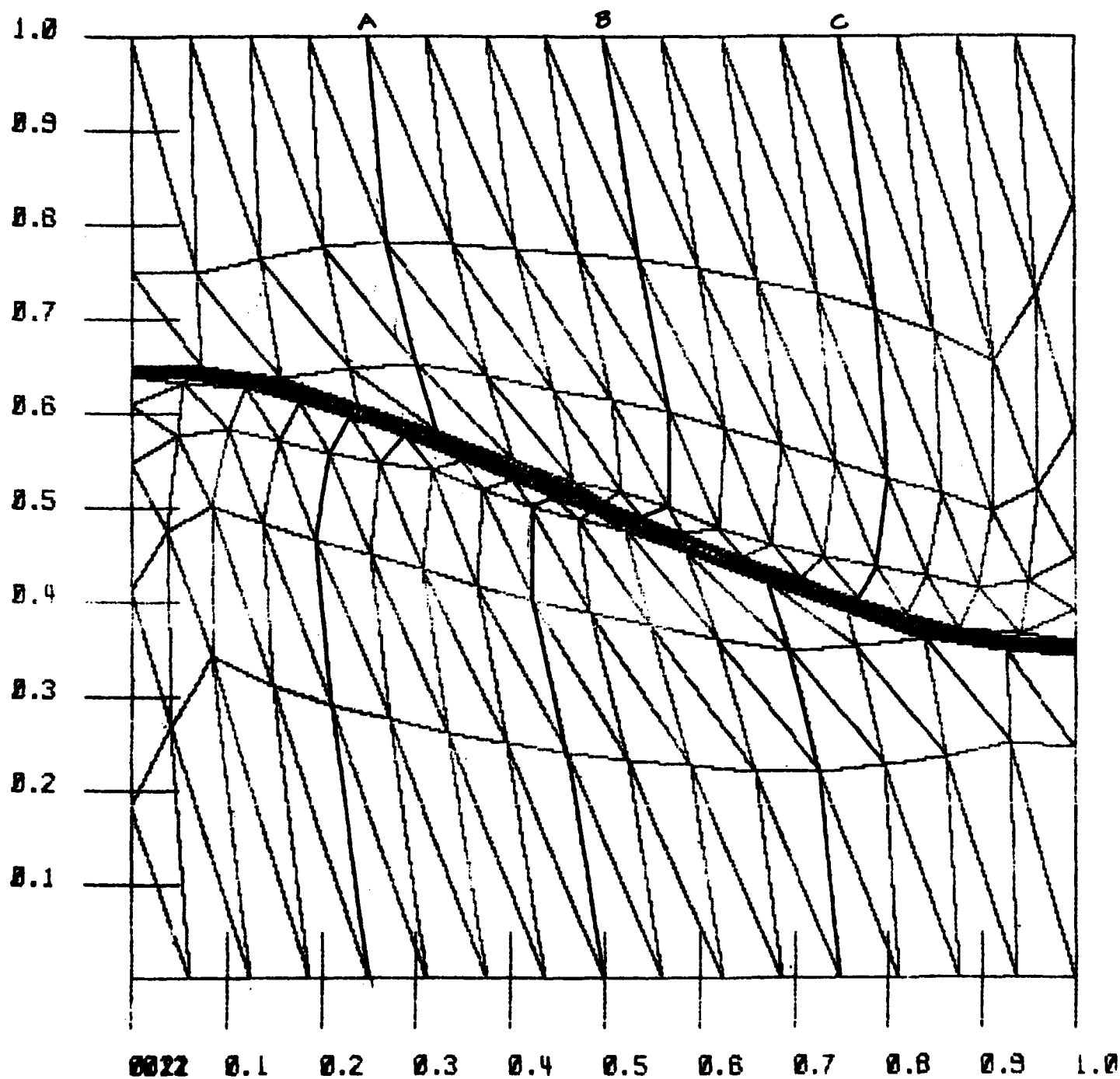


Figure 11. MFE grid projections on the x-y plane in the 2-D Burger-like example on a 16 x 16 grid with $\nu = 0.002$.

We have now barely opened a potentially vast area of investigation of regularization techniques for adaptive PDE meshes. This is certainly one of the key areas where we, and perhaps many others, can profitably expend greater efforts, particularly in view of the very attractive MFE properties which are emerging in: (i) alternative co-ordinate systems and (ii) the treatment of interface phenomena, both of which have a direct bearing on, and relationship to alternative regularization methods. Here also the MFE method is a certain kind of research pacesetter which is suggesting means of applying these new mathematical methods of regularizing PDE node motions in both new and conventional PDE methods so as to resolve such historically persistent dilemmas as singularities at origins of spherical and cylindrical coordinate systems, artificial smearing of interfaces, unduly constrained grid aspect ratios, numerical diffusion, severe time step constraints, and grid node utilization.

5. ALTERNATIVE CO-ORDINATE SYSTEMS

Initial work is in progress on 2-D MFE calculations in cylindrical co-ordinates. The results in this task also provide guidance for later developments in spherical co-ordinates. The major issue of present interest is the apparently natural elimination of singularities at the origin. Such singularities have historically plagued many conventional PDE methods.

Transport equations contain, in cylindrical co-ordinates, for example, advection operators of the form $\frac{1}{r} \frac{\partial}{\partial r} (ry)$, where r is the radial co-ordinate; and y is a dependent variable of the PDE system. Singularities or other anomalous features frequently occur in various discretized representations of the term (y/r) as $r \rightarrow 0$. In contrast, the MFE discretization is formulated in terms of well-defined inner products which eliminate such possible singularities. For example, the inner products of the term (y/r) with the basis function α , taken over the interval Δr , is given by

$$\int_{\Delta r} (y/r) \cdot \alpha \cdot r dr = \int_{\Delta r} y \cdot \alpha dr .$$

The integral of $\alpha \cdot y dr$ is essentially analytic and is readily evaluated everywhere on the problem domain. This attractive MFE property in cylindrical co-ordinates obviously holds in a similar manner in spherical co-ordinates. The properties of these r -weighted norms are naturally different than the MFE

inner products which were used in the Cartesian co-ordinate systems considered in earlier MFE work. Analysis and testing of these properties associated with r-weighted norms and of node controlling penalty functions in cylindrical co-ordinates have been undertaken in this period, and extensive work is expected to continue in this area in the future in order to understand and exploit fully the benefits of this analytic MFE formulation of otherwise troublesome PDE operators in cylindrical and spherical co-ordinates.

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4. Gelinas, R.J. and S.K. Doss, "The Moving Finite Element Method: A Semi-Automatic Solver for Diverse PDE Applications," Advances in Computer Methods for Partial Differential Equations--IV, pg. 230-239, edited by R. Vichnevetsky and R.S. Stepleman, Proceedings, Fourth IMACS International Symposium on Computer Methods for Partial Differential Equations, Lehigh University, Bethlehem, PA, June 30-July 2, 1981.
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PUBLICATIONS

Present Work Period (Year 2)

GELINAS, R.J., S.K. Doss, J.P. Vajk, J. Djomehri and K. Miller, "Moving Finite Elements in 2-D," Proceedings, Vol. 1, pg. 58-60, edited by R. Vichnevetsky, 10th IMACS World Congress on Systems Simulation and Scientific Computation, Montreal, Canada, August 8-13, 1982. (Refereed)

GELINAS, R.J. and S.K. Doss, "The Moving Finite Element Method: 1-D Transient Flow Applications," Proceedings, Vol. 1, pg. 156-158, edited by R. Vichnevetsky, 10th IMACS World Congress on Systems Simulation and Scientific Computation, Montreal, Canada, August 8-13, 1982. (Refereed)

Previous Work Period (Year 1)

GELINAS, R.J., S.K. Doss and K. Miller, "The Moving Finite Element Method: Applications to General Partial Differential Equations with Multiple Large Gradients," J. Comp. Phys., 40, No. 1, pg. 202, March 1981.

GELINAS, R.J. and S.K. Doss, "The Moving Finite Element Method: A Semi-Automatic Solver for Diverse PDE Applications," Advances in Computer Methods for Partial Differential Equations--IV, pg. 230-239, edited by R. Vichnevetsky and R.S. Stepleman, Proceedings, Fourth IMACS International Symposium on Computer Methods for Partial Differential Equations, Lehigh University, Bethlehem, PA, June 30-July 2, 1982. (Refereed)

PROZNITZ, D., R.A. Haas, S.K. Doss and R.J. Gelinas, "A Two-Dimensional Numerical Model of a Free Electron Laser," J. of Quantum Electronics, Vol. 9, p. 1047-69. (Summary also presented at the Conference on Lasers and Electro-Optics (CLEO'81), Washington, DC, June 10-12, 1981.)

Manuscripts in Preparation

Seven manuscripts are in various stages of preparation for journal submission, as indicated below:

Title	Authors	Journal
1. Analytic Properties of the MFE Method	S.K. Doss, N.N. Carlson, R.J. Gelinas	<u>SIAM</u>
2. Applications of The Moving Finite Element Method in 2-D	R.J. Gelinas, S.K. Doss, K. Miller, N.N. Carlson	<u>J.Comp.Phys.</u>
3. Real Versus Non-Physical Dissipation Processes in Shock Calculations	R.J. Gelinas, S.K. Doss, N.N. Carlson	<u>J.Applied Physics</u>
4. An O.D.E. Solver for PDE Applications	K. Miller, N.N. Carlson, S.K. Doss	<u>J.Comp.Phys.</u>
5. Irregular Reflection of Planar Shocks in 2-D	R.J. Gelinas, S.K. Doss, N.N. Carlson	<u>Physics of Fluids</u>
6. Applications of the MFE Method in Continuum Mechanics	R.J. Gelinas, S.K. Doss, N.N. Carlson	<u>J.Comp.Phys.</u>
7. Analysis of Alternative Basis Functions in the MFE Method	S.K. Doss, N.N. Carlson, R.J. Gelinas	<u>J.Comp.Phys.</u> or <u>SIAM</u>

PERSONNEL

The personnel associated with this research effort during this reporting period are:

Mr. Neil N. Carlson (Neil is a full-time SAI employee with a B.S. degree in Mathematics. He will enroll as a graduate student in Mathematics at U.C. Berkeley in Fall, 1983. He will probably pursue his Ph.D. degree as a student of Prof. Keith Miller.)

Dr. M. Jahed Djomehri (Student of Prof. Keith Miller; graduated with Ph.D.)

Dr. Said K. Doss

Dr. Robert J. Gelinas

Dr. J. Peter Vajk

Programmers (O. Ofiesh, S. Schell, and D. Robles)

INTERACTIONS

Coupling activities during this work period include:

Papers Presented at Meetings and Conferences

1. Gelinas, R.J., S.K. Doss, and N.N. Carlson, "Moving Finite Element Research for Shock Hydrodynamics, Continuum Mechanics, and Combustion," First Army Conference on Applied Mathematics and Computing, Sponsored by the Army Mathematics Steering Committee, George Washington University, Washington, DC, May 9-11, 1983.

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3. Gelinas, R.J., S.K. Doss, and N.N. Carlson, "A Comparative Study of Physical Versus Non-Physical Dissipation Processes in Reflected Shocks in Gases" Workshop on 1-D Hydrodynamics, Lawrence Livermore Laboratory, Livermore, CA, September 8-10, 1983.

4. Gelinas, R.J., S.K. Doss, J.P. Vajk, J. Djomehri and K. Miller, "Moving Finite Elements in 2-D," 10th IMACS World Congress on Systems Simulation and Scientific Computation, Montreal, Canada, August 8-13, 1982.

5. Gelinas, R.J. and S.K. Doss, "The Moving Finite Element Method: 1-D Transient Flow Applications," 10th IMACS World Congress on Systems Simulation and Scientific Computation, Montreal, Canada, August 8-13, 1982.

SEMINARS GIVEN ON OUR MFE RESEARCH

1. Air Force Weapons Laboratory (Major Raymond Bell)
2. Sandia Livermore Laboratory (Dr. R. Kee)
3. University of Toronto (Prof. I. Glass)
4. Lawrence Livermore Laboratory (H. Division Seminar)
5. Exxon Research (Dr. G. Byrne)

CONSULTIVE FUNCTIONS TO OTHER AGENCIES

1. Eglin Air Force Base; we are currently under contract to investigate the application of the MFE method in 2-D to armor penetration effects; contact: Major Guy Spitale.

2. Defense Nuclear Agency; we are currently under contract to investigate the application of the MFE method in 2-D to airblast effects; contact: Dr. George Ullrich.

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